STUDIES ON TODA MAP(Theory of Dynamical Systems and Singular Phenomena)

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ABSTRUCT

A nearly integrable Hamiltonian system with many degrees of freedom is studied. Especially, a numerical method extracting soliton components is presented. As an application, escape time in action space is calculated and its relation to system size is investigated.

§1. INTRODUCTION

Recent researches in dynamical systems has made a remarkable progress, especially in low-dimensional systems and dissipative systems.[1)] However, Hamiltonian systems with many degrees of freedom have not been studied extensively. In this paper we present a numerical study on a nearly integrable mapping, and discuss its behavior in action variable space.

Hamiltonian systems with many degrees of freedom concern two problems. First, as models in field theory, the stability of solutions should be discussed (i.e., KAM theory in infinitely many degrees of freedom). Second, as a foundation of classical statistical mechanics, measurement of Arnol'd diffusion and establishment of ergodicity are of great interest.

In both problems, it is useful to start with nearly integrable systems. Here we list three types of such systems with many degrees of freedom. Hereafter the particle number N will be often referred as the total system size.

1) weakly coupled local oscillators;

Hamiltonian of the system is, for example,[2)]

$$H = \sum_{j=1}^{N} \frac{1}{2} I_j^2 + \varepsilon \cos(\theta_j - \theta_{j+1}) \quad . \quad (1.1)$$

Systems of this type have a property that the connectance among variables is independent of N.

2) weakly perturbed linear chain;

Hamiltonian of the system is, for example,

$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \frac{1}{2} (x_j - x_{j+1})^2 + \varepsilon (x_j - x_{j+1})^4 \quad . (1.2)$$

As the integrable part is linear with respect to action variable, only one frequency can destroy all the solutions of unperturbed system.

3) weakly perturbed nonlinear integrable systems;

Hamiltonian of the system is, for example,

$$H = \sum_{j=1}^{N} \frac{1}{2}P_j^2 + \exp(Q_j - Q_{j+1}) + \varepsilon Q_j \cos(wt-pj) \quad .(1.3)$$

Toda map, which will be discussed in this paper, is one of these systems. Owing to the development of soliton theory, systems of this type can be cast into action-angle systems[3] (of unperturbed systems).

We introduce "Toda map" in §2. Then we show how to calculate action variables in §3. In §4 , we will see how perturbed mapping behaves in the space of action variables. The notion of escape time is introduced in §5, where numerical result is given. The last section is devoted to summary and discussion.

§2. TODA MAP

Toda map[4] , discrete time equation of Toda lattice, is a mapping defined on (1+1)-dimensional discrete spacetime. It reads as

$$\exp(Q_n^{m+1} - Q_n^m) - \exp(Q_n^m - Q_n^{m-1})$$

$$= \delta^2 (\exp(Q_{n-1}^m - Q_n^m) - \exp(Q_n^m - Q_{n+1}^m)) \qquad , (2.1)$$

where δ is a real constant which corresponds to temporal spacing, and Q_n^m is a real valued field with integer indices m (time) and n (space).

Introducing the momentum P_n^m as

$$P_n^m \equiv [\exp(Q_n^m - Q_n^{m-1}) -1]/\delta \qquad , \qquad (2.2)$$

we obtain

$$P_n^{m+1} = P_n^m + \delta(\exp(Q_{n-1}^m - Q_n^m) - \exp(Q_n^m - Q_{n+1}^m)), \quad (2.3)$$

which , in the limit $\delta \to 0$, reduces to the equation of motion of the Toda lattice,[6]

$$\frac{dP_n}{dt} = \exp(Q_{n-1} - Q_n) - \exp(Q_n - Q_{n+1}) \qquad . \quad (2.4)$$

Toda map (2.2) is a Hamiltonian system ; that is, its time evolution is a canonical transformation

$$\sum_n dP_n^{m+1} \wedge dQ_n^{m+1} = \sum_n dP_n^m \wedge dQ_n^m \quad . \qquad (2.5)$$

Moreover, it is a completely integrable system. Toda map with N degerees of freedom possesses N independent conserved quantities. We will see it in a later section.

Similarly to the (continuous-time) Toda lattice, Toda map has some special solutions. In the infinite chain, 1-soliton solution of Toda map reads as[4)]

$$\exp(\ Q_n^m\ ) = \frac{1 + \exp\{\ 2[\ \omega(m+1) - p(n-1)\ ]\ \}}{1 + \exp\{\ 2[\ \omega m - pn\ ]\ \}} \quad , \qquad (2.7)$$

with dispersion relation

$$\sinh(\ \omega\ ) = \pm \sinh(\ p\ )\delta \quad . \qquad (2.8)$$

And in the case of periodic chain with period N , we have[6)]

$$\exp(\ Q_n^m\ ) = \frac{\theta_0\{\ g[\omega(m+1) - p(n-1)\ ]\ \}}{\theta_0\{\ g[\omega m - pn\ ]\ \}} \quad , \qquad (2.9)$$

with

$$\theta_1(\ g\ \omega) = \pm\ \theta_1(\ g\ p\ )\delta \quad , \qquad (2.10)$$

and

$$g \equiv 1/pN \qquad . \qquad (2.11)$$

See Appendix A for detail.

§3. FLOQUET THEORY

We consider the system with periodic boundary condition. In order to derive conserved quantities and calculate action variables, we construct a one parameter function $\Delta^m(z)$, which is called " Floquet discriminant".

First we consider the following recursion relation for sequence ψ_n:

$$a_n^m \, \psi_{n+1}^m + b_n^m \psi_n^m + a_{n-1}^m \psi_{n-1}^m = \mu \qquad , \qquad (3.1)$$

where the coefficients a, b, and parameter μ are defined as

$$a_n^m \equiv \exp[\ (Q_n^m - Q_{n+1}^m)/2\ ] \qquad , \ (3.2a)$$

$$b_n^m \equiv \frac{P_n^m - [\ \exp(\ Q_n^m - Q_{n+1}^m) + 1\ ]\delta}{[\ (1+\delta z)(1+\delta z^{-1})\ ]^{1/2}} \qquad , \ (3.2b)$$

$$\mu \equiv \frac{z + 1/z}{[\ (1+\delta z)(1+\delta z^{-1})\ ]^{1/2}} \quad , \qquad (3.2c)$$

and the parameter z varies in the following regions;

$$-1 \leqq z < -d, \quad 0 < z \leqq 1 \quad ,$$
$$\text{and} \quad z = \exp(\ i\phi\ )\ , \ 0 \leqq \phi < 2\pi. \qquad (3.3)$$

As eq.(3.1) is a linear recursion relation among three ψ's, the space of its solution is two dimensional. Hence, given two independent solutions of eq. (3.1) , say ϕ and $\overline{\phi}$, then any solution of eq.(3.1) can be expressed as linear combination of ϕ and $\overline{\phi}$. Therefore , for a periodic lattice of N particles, there exist constants α, β, $\overline{\alpha}$, $\overline{\beta}$ (which may depend on "time" m ,but does not on "space" n .) such that

$$\psi^m_{N+1}(z) = \alpha^m(z)\psi^m_1(z) + \beta^m(z)\overline{\psi}^m_1(z) \quad , \qquad (3.4a)$$

$$\overline{\psi}^m_{N+1}(z) = \overline{\beta}^m(z)\psi^m_1(z) + \overline{\alpha}^m(z)\overline{\psi}^m_1(z) \quad . \qquad (3.4b)$$

Or, in a matrix form,

$$\Psi^m_{N+1}(z) = \Psi_1^m(z) M^m(z) \quad , \qquad (3.5)$$

where

$$\Psi_n^m(z) \equiv \begin{pmatrix} \psi_n^m(z) & \overline{\psi}_n^m(z) \\ \psi_{n+1}^m(z) & \overline{\psi}_{n+1}^m(z) \end{pmatrix} , \quad (3.6a)$$

and

$$M^{m}(z) \equiv \begin{pmatrix} \alpha^m(z) & \overline{\beta}^m(z) \\ \beta^m(z) & \overline{\alpha}^m(z) \end{pmatrix} . \quad (3.6b)$$

Floquet discriminant $\Delta(z)$ is defined as

$$\Delta^m(z) = \mathrm{trace}(M^m(z)) . \quad (3.7)$$

A remarkable feature of $\Delta(z)$ is that it is independent of "time" m under pure Toda map ;

$$\Delta^{m+1}(z) = \Delta^m(z) \quad \text{for all } z , \quad (3.8)$$

which can easily be shown by a straightforward calculation.

The space-time profile and Floquet spectrum of the above solution are shown in Fig.1.

Expanding $\Delta(z)$ as

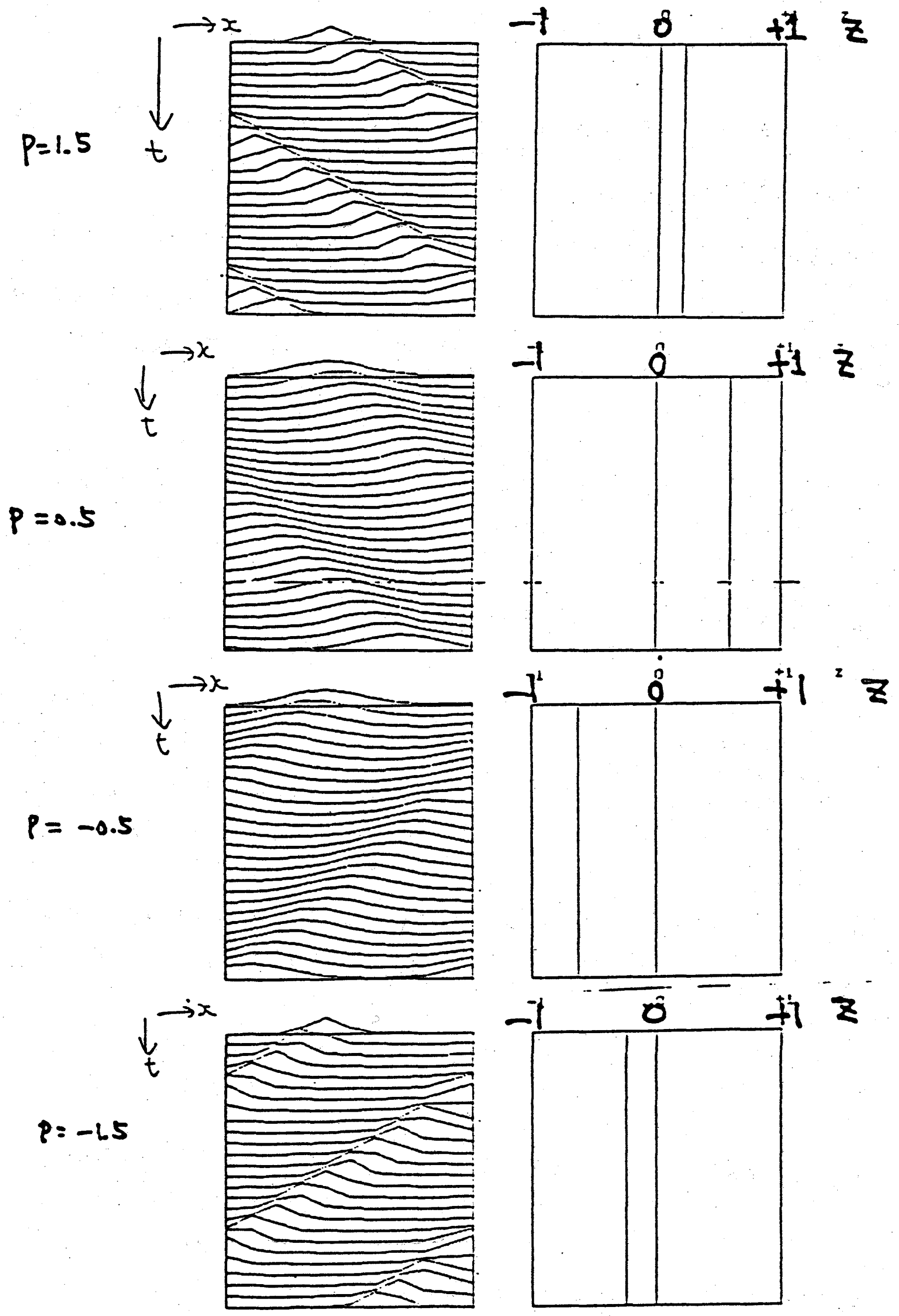

Fig.1 Temporal evolution of real space configuration $Q_n - Q_{n+1}$ (left), and zeros of Floquet discriminant (right). Initial condition is given by 1-soliton solution (2.7). Note the dependence of z on p.

$$\Delta^m(z) = \sum_{j=-N}^{N} C_j \lambda^j \quad , \quad \lambda \equiv z+1/z \quad , \quad (3.9)$$

we see that all C_j's are constant of motion , i.e., functions of action variables only.

From the Floquet discriminant Δ, we can calculate action variables. When $|\delta| \ll 1$ and $N \gg 1$, the action variables of solitons are approximately written as[3)]

$$I_j = z_j + 1/z_j \quad , \quad (3.10)$$

where z_j is a zero of $\Delta(z)$;

$$\Delta(z_j) = 0 \quad . \quad (3.11)$$

Note that, because of eq.(3.8), z_j itself is invariant for any δ, N .

Making use of these procedures numerically, we can poject perturbed system on "soliton basis".[7)]

§4. GENERAL FEATURES OF THE PERTURBED SYSTEM

We shall consider the following map :

$$P_n^{m+1} = P_n^m + \delta\left(\exp(Q_{n-1}^m - Q_n^m) - \exp(Q_n^m - Q_{n+1}^m) + a\cos(wm - qn) \right) \quad . \quad (4.1)$$

In the limit $\delta \to 0$, this system can be derived from Hamiltonian

$$H = H_0 + aH_1 \quad , \quad (4.2a)$$

$$H_0 = \sum_n \exp(Q_n - Q_{n+1}) + \frac{1}{2}P_n^2 \quad , \quad (4.2b)$$

$$H_1 = -\sum_n Q_n \cos(wm - qn) \quad . \quad (4.2c)$$

An example of time evolution of eq.(4.1) is shown in Fig.2, with a=0.1(rather strong perturbation) , where real space profile and zeros of Floquet discriminant (= soliton parameters) are shown. We can see many solitons emerging from the edge z = ± 1. This phenomenon can be interpreted as follows. In most soliton systems, action variables fall into two classes; one corresponds to extended modes (which, in infinite chain, form continuous spectrum on parameter z) and the other corresponds to localized modes (which, in infinite chain, form discrete spectrum). We take the case of Toda lattice;

$$\sum_n dP_n \wedge dQ_n = \int dI(z) \wedge d\theta(z) + \sum_j dI_j \wedge d\theta_j , \quad (4.3a)$$

$$z = \exp(i\phi) \quad , \quad (4.3b)$$

$$0 \leqq \phi < 2\pi \quad . \quad (4.3c)$$

The time derivative of angle variables vanishes at z = ±1, where

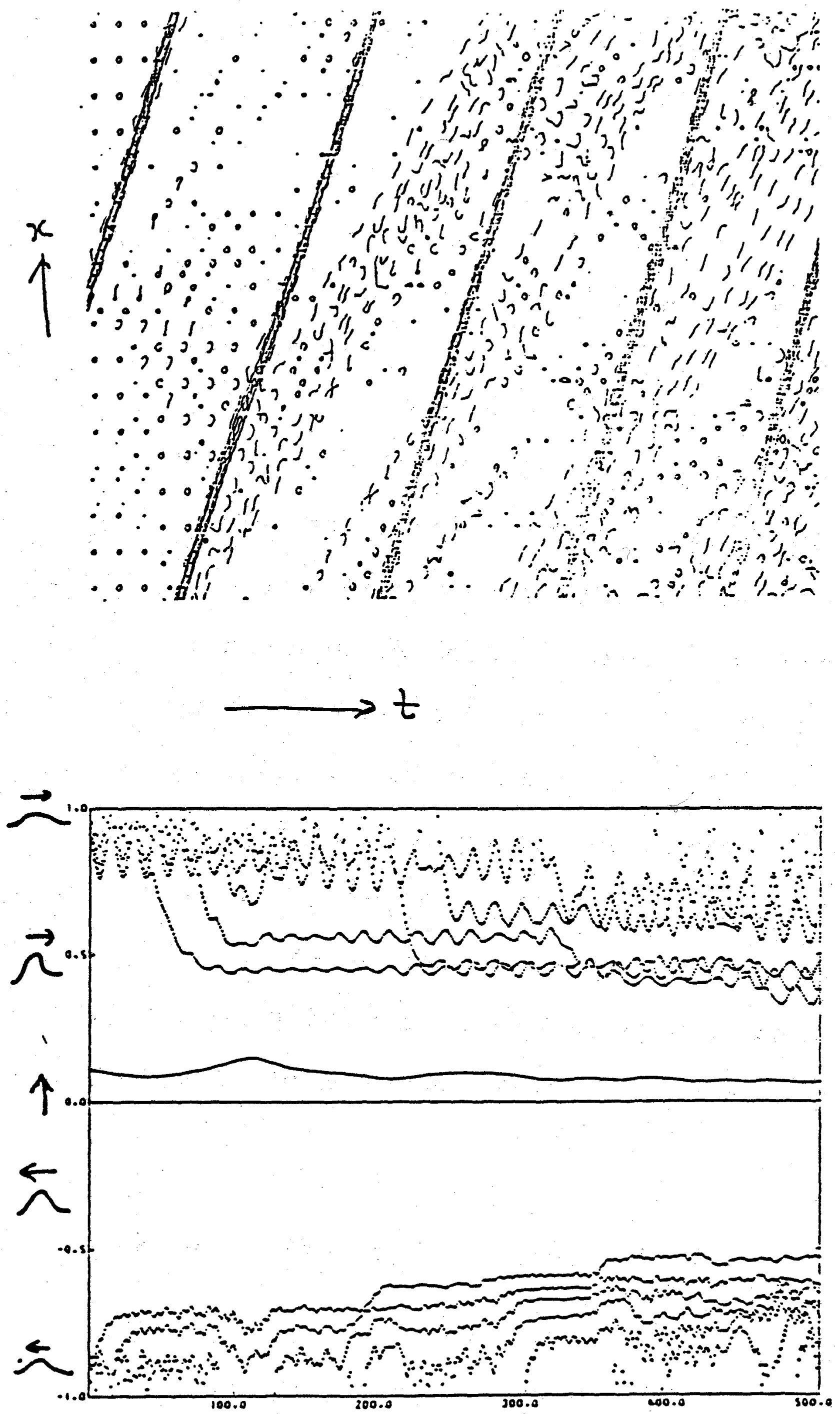

Fig.2 Temporal evolution of real space configuration (above) and zeros of Δ (below) for the map (4.1), with a = 0.1, d = 0.05, N = 50.

the two modes (extended ones and localised ones) coincide.

$$\frac{d}{dt}\theta_j = z_j - z_j^{-1} \quad , \quad (4.4a)$$

$$\frac{d}{dt}\theta(\exp(i\phi)) = 2\sin(\phi) \quad , \quad (4.4b)$$

$$\frac{d}{dt}\theta_j \Big|_{z=\pm 1} = \frac{d}{dt}\theta(\exp(i\phi))\Big|_{\phi=0,\pi} = 0 \ . (4.4c)$$

Thus, the points $z=\pm 1$ are structurally fragile against perturbation. We point out that emergence and growth of solitons at the boundary of two modes are the most important feature in perturbed soliton systems.

§5. ESCAPE TIME IN ACTION SPACE

Recently, Hamiltonian systems with N=1 are extensively studied.[1)] Boundedness of their motion is completely governed by the existence of KAM torus, which devides the 2-dimensional phase space into two disconnected regions. So it is important to examine whether KAM torus exist or not, and several criterions have been reported.

On the other hand, for $N > 1$, all the regions except for invariant sets are connected even if KAM torus exists, because an N-dimensional torus cannot devide the whole 2N (or 2N-1, if energy is conserved) phase space into two regions. Thus, a general Hamiltonian system with many degrees of freedom is expected to voyage over the whole phase space, and finally attain ergodicity. This phenomenon is called "Arnol'd diffusion".[8),9)]

It is known that the speed of Arnol'd diffusion is quite slow[10)]. In order to accomplish thrmal equilibrium, the speed should become faster as the degree of freedom increases. Otherwise we would not observe a thermal equilibrium state, as the total volume of phase space gets exponentially large with the increase of N.

In order to confirm that most macroscopic Hamiltonian systems establish thermal equilibrium within a realistic time (which is much shorter than Poincare's recurrance time), we measure the velocity in the action space, or equivalently, "escape time " in action space.

We calculate the escape time (defined later) for the mapping

$$P_n^{m+1} = P_n^m + \delta(\ \exp(\ Q_{n-1}^m - Q_n^m) - \exp(\ Q_n^m - Q_{n+1}^m)\ \ ) + \begin{cases} \delta a\ \cos(wm - pn) & , \quad 1 \leqq n \leqq N_f\ , \\ 0 & , \text{ otherwise} \end{cases} \quad , \tag{5.1}$$

where N_f is the number of particles which are subject to external force applied, and N is the total system size. Our interest lies in the N-dependence of escape time T . We compare T's of various N's with N_f fixed so that we do not change the external effect. (Another possibility is to vary N_f proprotional to N .)

Escape time is defined as the minimal time steps that a system get away a fixed amount of distance from the initial point in action space. In order to make calculation easier, we take only soliton components among all action variables. This definition is legitimated because main contribution to the distance is the excitation of solitons from radiations (as mentined in §4) , and action variables of radiation modes are usually small (At most 1/N, compared to the action variables of soliton modes).

First we define

$$\begin{aligned} d\,(m) &\equiv \text{(numbers of solitons with negative velocities)}, \\ T &\equiv \min(m) \text{ such that } d(m) = 2 \end{aligned} \qquad . \text{ (5.2)}$$

Here we take into account only the solitons with negative velocities because the external force supplies positive momentum, and we would like to select diffusive component in the behavior of our system.

The result for N=40 to N=320, N_f =8, a=0.1~0.0125 is shown in Fig.3, where power-law dependence of escape time T on N

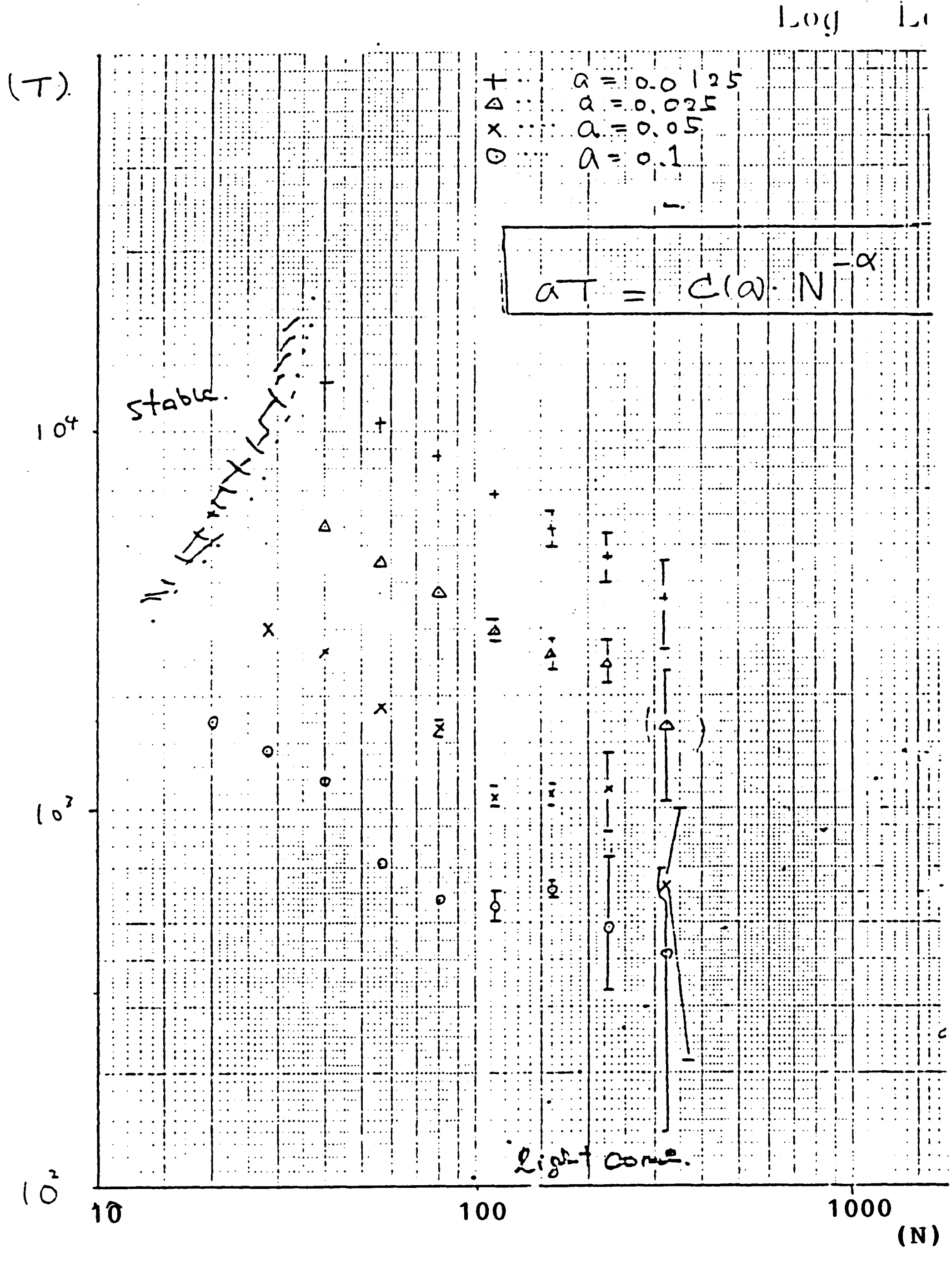

Fig.3 Escape time T and total system size N for the map (5.1). Here T is defined as $T \equiv \min(m)$ such that $d(m) > 1$.

$$T \sim \frac{1}{a} N^{-\alpha} \quad , \quad \alpha \sim 0.6 \quad , \tag{5.3}$$

is explicitly shown.

But, if we define

$$d_2(m) \equiv \sum_{j=1}^{\#} \left| I_j^m - I_j^0 \right| \quad , \tag{5.4}$$

where # is the number of all solitons at step m , then a numerical data shows that the escape time increases as N gets larger.

§6. SUMMARY AND DISCUSSION

We presented a method to study perturbed soliton systems using a "soliton basis". As an application, we obtained a numerical result that , in a case, the escape time decreases as a power of the total system size. This may be a reflection of the fact that as the system size gets large, the resonance structure becomes finer. If so, some renormalization method can be developed. But the result is not yet definitive , and further study is necessary.

Recently, similar power-law behavior has been observed in the coupled standard map.[11)] Defining the distance as

$$d(m) \equiv \max_j \left| I_j^m - I_j^0 \right| \quad , \tag{6.1}$$

the escape time varies as

$$T \sim 1/\sqrt{N} \quad . \qquad (6.2)$$

This can be interpreted as a diffusion of N independent variables.

Although we presented only numerical examples, the projection method may be applied to the analysis of experimental data.

Some questions remain unsolved. Beside "escape time", there is another interesting quantity, "thermalization time". It is defined as the time that a system establishes thermal equilibrium. We should find a relation between these two quantities. It is an interesting conjecture that glassy systems may be characterised as systems with infinite thermalization time.

For a long time since their discovery, solitons have attracted much attention because of their extreme stability. However, perturbed soliton systems exihibit more rich behavior such as creation and annihilation of solitons. The projection method on soliton basis, described in this paper, will serve as a useful method in analysing perturbed soliton systems, thus contributing to the study of nonlinear systems with infinite degrees of freedom.

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APPENDIX A. PERIODIC 1-SOLITON SOLUTION

In the equalities (2.9) and (2.10), θ_0 and θ_1 are Jacobi's elliptic functions, which are defined as

$$\theta_0(u) \equiv \frac{1}{\sqrt{g}} \sum_{n=-\infty}^{\infty} \exp\{ -\frac{\pi}{g}[u-(n+\frac{1}{2})]^2\} \quad , \tag{A.1}$$

$$\theta_1(u) \equiv \frac{1}{\sqrt{g}} \sum_{n=-\infty}^{\infty} (-1)^n \exp\{ -\frac{\pi}{g}[u-(n+\frac{1}{2})]^2\} \quad . \tag{A.2}$$

The function θ_0 has a periodicity

$$\theta_0(u+1) = \theta_0(u) \quad , \tag{A.3}$$

so the constant g is determined as eq.(2.11).

REFERENCES

1) As a review article, see, for example,
(dissipative systems)
E.Ott, Rev.Mod.Phys. 53 (1981) 655 , and
(Hamiltonian systems)
A.J.Lichtenberg and M.A.Lieberman, Regular and Stochastic Motion (Springer,New York,1983)

2) C.E.Wayne, Com.Math.Phys. 104 (1986) 21

3) For example, D.W.McLaughlin, J.Math.Phys. 16(1975) 96

4) R.Hirota, J.Phys.Soc.Jpn. 45(1978)321

5) M.Toda, Physics Reports 18C(1975) 1

6) M.Wadati, private communication

7) This method was originally developed by K.Fessar et.al., Phys.Rev.A31(1985) 2728 and see the references there in. See also E.A.Overman II, D.W.McLaughlin and A.R.Bishop, Physica 19D(1986)1

8) V.I.Arnol'd, Soviet Math.Dokl.5(1964)581

9) K.Kaneko and R.J.Bagley, Phys.Letts. 110A (1985) 435

10) N.N Nekhoroshev, Russ.Math.Surveys 32 (1977) 1

11) K.Kaneko, private communication